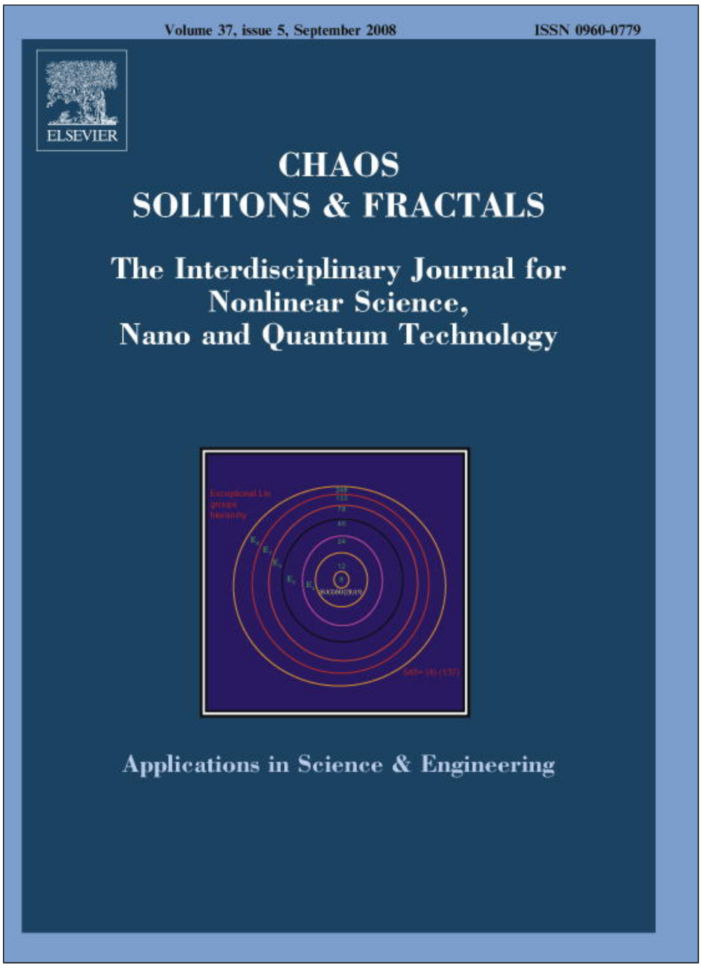
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Effect of vertical high-frequency parametric excitation on self-excited motion in a delayed van der Pol oscillator

Simohamed Sah, Mohamed Belhaq *

Laboratory of Mechanics, University Hassan II-Aïn Chock, PB 5366 Maârif, Casablanca, Morocco

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Abstract

We investigate the interaction effect of fast vertical parametric excitation and time delay on self-oscillation in a van der Pol oscillator. We use the method of direct partition of motion to derive the main autonomous equation governing the slow dynamic and then we apply the averaging technique on this slow dynamic to derive a slow flow. In particular we analyze the slow flow to analytically approximate regions where self-excited vibrations can be eliminated. Numerical integration is performed and compared to the analytical results showing a good agreement for small time delay. It was shown that vertical parametric excitation, in the presence of delay, can suppress self-excited vibrations. These vibrations, however, persist for all values of the excitation frequency in the case of a fast vertical parametric excitation without delay [Bourkha R, Belhaq M. Effect of fast harmonic excitation on a self-excited motion in van der Pol oscillator. Chaos, Solitons & Fractals, 2007;34(2):621–7.].

1. Introduction

In this paper, we study the interaction effect of a vertical fast harmonic (FH) parametric excitation and time delay on self-excited vibration in a van der Pol oscillator. Great attention has been paid in the last decade to the study of non-trivial effects of FH excitation on mechanical systems. Blekhman [1] developed a technique called the direct partition of motion based on splitting the dynamic into fast and slow motions. This method provides an approximation for the small fast dynamic and the main equation governing the averaged slow dynamic. A number of studies used this mathematical tool to analyze properties and dynamics of some mechanical systems. Tcherniak and Thomsen [2] studied slow dynamic effects of FH excitation for some elastic structures. Fidlin and Thomsen [3] examined asymptotic properties of systems with strong high-frequency excitation. Thomsen [4] analyzed some general effects of strong high-frequency excitation. The problem of quenching friction-induced oscillations for the mass-on-moving-belt model using FH external excitation has also been considered [5]. Bourkha and Belhaq [6] examined the effect of horizontal and vertical FH parametric excitation on periodic self-excited motion in a van der Pol pendulum. It was shown that the limit cycle can

* Corresponding author. Tel.: +212 897993; fax: +212 22230674.
E-mail address: mbelhaq@yahoo.fr (M. Belhaq).

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disappear by horizontal FH excitation for a certain critical value of the excitation frequency and persists in the case of a vertical FH excitation.

The purpose of the present work is to investigate the effect of vertical FH excitation, in the presence of time delay, on the persistence of the limit cycle in van der Pol oscillator.

Note that delayed parametrically excited oscillations has been considered by Insperger and Stepan [7]. They studied the stability chart for the delayed linear Mathieu equation using the method of exponential multipliers. Maccari [8] examined the parametric resonance of a van de Pol oscillator under a time delay state feedback. Nana Nbendjo and Woafo [9] considered the control with delay of an undamped buckled beam, subjected to parametric excitations. Ji and Hansen [10] studied the existence of limit cycle in a van der Pol-Duffing oscillator with time delay.

To analyse the effect of vertical FH parametric excitation and time delay on the suppression of limit cycle in van der Pol oscillator, we apply, in a first step, the method of direct partition of motion to derive the main autonomous equation governing the slow dynamic of the oscillator. The method of averaging is then performed on the slow dynamic to obtain a slow flow. The analysis of this slow flow provides analytical approximations of a region in parameter space where limit cycle can be eliminated. This work is an extension of a previous paper [6] in which the delay was omitted.

2. Direct partition of motion

Vibrations of pendulum with delay subjected to a vertical parametric forcing and to a self-excitation can be described in non-dimensional form by the following equation

$$\frac{d^2x}{dt^2} - (\alpha - \beta x^2)\frac{dx}{dt} + \sin x = a\Omega^2 \sin x \cos \Omega t + \lambda x(t - T) \tag{1}$$

where the parameters α and β are assumed to be small, a is the excitation amplitude, Ω is the parametric excitation frequency, and the parameters λ and T are the amplitude of the delay and the delay period, respectively. Eq. (1) has relevance to regenerative effect in high-speed milling. High-speed milling can induce a rapid parametric excitation and milling can generate self-oscillations; see for instance [11]. Note that this equation contains two characteristic frequencies, the frequency of periodic self-oscillations produced by the van der Pol component and the parametric excitation frequency Ω.

The goal here is to investigate the interaction effect of vertical high-frequency excitation Ω and time delay on the periodic motion (limit cycle) of Eq. (1). We focus our analysis on small vibrations around $x = 0$ by expanding in Taylor's series up to the third-order the term $\sin x \simeq x - \delta x^3$ where the coefficients $\delta = 1/6$. Eq. (1) becomes

$$\frac{d^2x}{dt^2} - (\alpha - \beta x^2)\frac{dx}{dt} + x - \delta x^3 = a\Omega^2(x - \delta x^3)\cos \Omega t + \lambda x(t - T) \tag{2}$$

To implement the method of direct partition of motion [1], we introduce two different time-scales: a fast time $T_0 \equiv \Omega t$ and a slow time $T_1 \equiv t$, and we split up $x(t)$ into a slow part $z(T_1)$ and a fast part $\epsilon\varphi(T_0, T_1)$ as follows

$$x(t) = z(T_1) + \epsilon\varphi(T_0, T_1) \tag{3}$$

and

$$x(t - T) = z(T_1 - T) + \epsilon\varphi(T_0 - \Omega T, T_1 - T) \tag{4}$$

where z describes slow motions at time-scale of oscillations of the pendulum, and $\epsilon\varphi$ stands for an overlay of the fast motions. Note that the slow motions z describes the main dynamics of the system. In Eqs. (3) and (4), ϵ indicates that $\epsilon\varphi$ is small compared to z. Since Ω is considered as a large parameter we choose $\epsilon \equiv \Omega^{-1}$, for convenience. The fast part $\epsilon\varphi$ and its derivatives are assumed to have a zero T_0-average, so that $\langle x(t)\rangle = z(T_1)$ and $\langle x(t - T)\rangle = z(T_1 - T)$ where $\langle\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi}()\,dT_0$ defines time-averaging operator over one period of the fast excitation with the slow time T_1 fixed.

Inserting (3) and (4) into (2) and introducing $D_i^j \equiv \frac{\partial^j}{\partial^j T_i}$ yields

$$\begin{aligned} &D_1^2 z + \epsilon D_1^2\varphi + 2D_0D_1\varphi + \epsilon^{-1}D_0^2\varphi - \alpha(D_1z + \epsilon D_1\varphi + D_0\varphi) + \beta(z^2D_1z + \epsilon z^2D_1\varphi + z^2D_0\varphi + 2\epsilon z\varphi D_1z + 2\epsilon z\varphi D_0\varphi) \\ &\quad + z + \epsilon\varphi - \delta(z^3 + 3\epsilon z^2\varphi) = \epsilon^{-1}(a\Omega)z\cos T_0 - \epsilon^{-1}(a\Omega)\delta z^3\cos T_0 + (a\Omega)\varphi\cos T_0 - 3\epsilon(a\Omega)\delta z\varphi^2\cos T_0 \\ &\quad - 3\epsilon(a\Omega)\delta z^2\varphi\cos T_0 + \lambda z(T_1 - T) + \epsilon\lambda\varphi(T_0 - \Omega T, T_1 - T) \end{aligned} \tag{5}$$

Averaging (5) leads to

$$D_1^2z - \alpha D_1z + \beta z^2D_1z + z - \delta z^3 = (a\Omega)\langle\varphi\cos T_0\rangle - 3\epsilon(a\Omega)\delta z^2\langle\varphi\cos T_0\rangle - 3\epsilon(a\Omega)\delta z\langle\varphi^2\cos T_0\rangle + \lambda z(T_1 - T) \tag{6}$$

Subtracting (6) from (5), an approximate expression for $\epsilon\varphi$ is obtained by considering only the dominant terms of order ϵ^{-1} as

$$D_0^2\varphi = a\Omega(z - \delta z^3)\cos T_0 \tag{7}$$

where it is assumed that $a\Omega = O(\epsilon^0)$. The stationary solution to the first order for φ is written as

$$\varphi = -a\Omega(z - \delta z^3)\cos T_0 \tag{8}$$

Retaining the dominant terms of order ϵ^0 in Eq. (6), inserting φ from (8) and using that $\langle \cos^2 T_0 \rangle = 1/2$ gives

$$D_1^2 z - (\alpha - \beta z^2)D_1 z + z - \delta z^3 + \frac{(a\Omega)^2}{2}(1 - 3\delta z^2)(z - \delta z^3) = \lambda z(T_1 - T) \tag{9}$$

The autonomous Eq. (9) governs the slow dynamic of the motion in the presence of the excitation frequency Ω. It includes the van der Pol component that produces a limit cycle, and hence, the effect of high-frequency excitation Ω on the limit cycle can be investigated through analytical predictions.

3. Averaging and slow flow

We examine the effect of the high-frequency excitation on the limit cycle in the slow dynamic (9) by performing the averaging method. We introduce a small parameter μ and we scale parameters $\alpha = \mu\tilde{\alpha}$, $\beta = \mu\tilde{\beta}$, $\delta = \mu\tilde{\delta}$ and $\lambda = \mu\tilde{\lambda}$. Then, neglecting nonlinear terms of fifth order, Eq. (9) reads

$$\ddot{z} - \mu(\tilde{\alpha} - \tilde{\beta}z^2)\dot{z} + \omega_0^2 z - \mu\tilde{\delta}(1 + 2(a\Omega)^2)z^3 = \mu\tilde{\lambda}z(t - T) \tag{10}$$

where $\dot{z} = \frac{dz}{dt}$ and $\omega_0^2 = \left(1 + \frac{(a\Omega)^2}{2}\right)$. In the case $\mu = 0$, Eq. (10) reduces to

$$\ddot{z}(t) + \omega_0^2 z(t) = 0 \tag{11}$$

with the solution

$$z(t) = R\cos(\omega_0 t + \phi), \quad \dot{z}(t) = -\omega_0 R\sin(\omega_0 t + \phi) \tag{12}$$

For $\mu > 0$, a solution is sought in the form (12) but R and ϕ are treated as time dependent. Variations of parameters gives the following equations on $R(t)$ and $\phi(t)$:

$$\dot{R}(t) = -\frac{\mu}{\omega_0}\sin(\omega_0 t + \phi)F(R\cos(\omega_0 t + \phi), -\omega_0 R\sin(\omega_0 t + \phi), t) \tag{13}$$

$$\dot{\phi}(t) = -\frac{\mu}{\omega_0 R}\cos(\omega_0 t + \phi)F(R\cos(\omega_0 t + \phi), -\omega_0 R\sin(\omega_0 t + \phi), t) \tag{14}$$

where

$$F(z, \dot{z}, t) = (\tilde{\alpha} - \tilde{\beta}z^2)\dot{z} + \tilde{\delta}(1 + 2(a\Omega)^2)z^3 + \tilde{\lambda}z(t - T) \tag{15}$$

with $z(t)$ is given by (12).

Using the averaging method [12,13] for small μ and replacing the right-hand sides of (13) and (14) by their averages over one 2π period, since Eq. (10) is autonomous, we obtain:

$$\dot{R} \approx -\frac{\mu}{\omega_0}\frac{1}{2\pi}\int_0^{2\pi}\sin(\omega_0 t + \phi)F\,dt \tag{16}$$

$$\dot{\phi} \approx -\frac{\mu}{\omega_0 R}\frac{1}{2\pi}\int_0^{2\pi}\cos(\omega_0 t + \phi)F\,dt \tag{17}$$

in which

$$F = -(\tilde{\alpha} - \tilde{\beta}R^2\cos(\omega_0 t + \phi)^2)\omega_0 R\sin(\omega_0 t + \phi) + \tilde{\delta}(1 + 2(a\Omega)^2)R^3\cos(\omega_0 t + \phi)^3 + \tilde{\lambda}\tilde{R}\cos(\omega_0 t - \omega_0 T + \tilde{\phi}) \tag{18}$$

with $\tilde{R} = R(t - T)$ and $\tilde{\phi} = \phi(t - T)$. Evaluating the integrals in (16) and (17) yields

$$\dot{R} = \mu\left(\frac{\tilde{\alpha}}{2}R - \frac{\tilde{\beta}}{8}R^3 - \frac{\tilde{\lambda}}{2}\frac{\tilde{R}}{\omega_0}\sin(\omega_0 T - \tilde{\phi} + \phi)\right) \tag{19}$$

$$\dot{\phi} = \mu\left(-\frac{3\tilde{\delta}}{8}\frac{R^2}{\omega_0}(1 + 2(a\Omega)^2) - \frac{\tilde{\lambda}}{2}\frac{\tilde{R}}{\omega_0 R}\cos(\omega_0 T - \tilde{\phi} + \phi)\right) \tag{20}$$

Eqs. (19) and (20) show that $\dot{R}$ and $\dot{\phi}$ are $O(\mu)$. We now expand in Taylor's series $\tilde{R}$ and $\tilde{\phi}$:

$$\tilde{R} = R(t-T) = R(t) - T\dot{R}(t) + \frac{1}{2}T^2\ddot{R}(t) + \cdots \tag{21}$$

$$\tilde{\phi} = \phi(t-T) = \phi(t) - T\dot{\phi}(t) + \frac{1}{2}T^2\ddot{\phi}(t) + \cdots \tag{22}$$

Then, we can replace $\tilde{R}$ and $\tilde{\phi}$ by $R(t)$ and $\phi(t)$ in Eqs. (19) and (20) since $\dot{R}$ and $\dot{\phi}$ and $\ddot{R}$ and $\ddot{\phi}$ are of $O(\mu)$ and $O(\mu^2)$, respectively [14]. This approximation reduces the infinite dimensional problem into a finite dimensional one by assuming μT is small.

After substituting the above approximation into (19) and (20), we obtain the following slow flow of the slow dynamic of motion

$$\dot{R} = \left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\omega_0 T\right)R - \frac{\beta}{8}R^3 \tag{23}$$

$$\dot{\phi} = -\frac{\lambda}{2\omega_0}\cos\omega_0 T - \frac{3\delta}{8\omega_0}(2(a\Omega)^2 + 1)R^2 \tag{24}$$

4. Equilibria and self-excited oscillations

Note that an equilibrium point in Eqs. (23) and (24) corresponds to a periodic motion in the original system (10). Equilibria are obtained by setting $\dot{R} = \dot{\phi} = 0$ in Eqs. (23) and (24). This leads to the two equilibria

$$R = 0, \quad R = \sqrt{\frac{8}{\beta}\left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\omega_0 T\right)} \tag{25}$$

The solution $R = \sqrt{\frac{8}{\beta}\left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\omega_0 T\right)}$ corresponding to a periodic motion (limit cycle) must be real. This leads to the inequality

$$\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\omega_0 T \geqslant 0 \tag{26}$$

By setting $\omega_0 = \sqrt{1 + \frac{(a\Omega)^2}{2}}$, Eq. (26) becomes

$$\sin\left(\sqrt{1 + \frac{(a\Omega)^2}{2}}T\right) \leqslant \frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}} \tag{27}$$

The above inequality provides the two following conditions corresponding to the birth of the limit cycle

$$T < \frac{1}{\sqrt{1 + \frac{(a\Omega)^2}{2}}}\arcsin\left(\frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}}\right) \tag{28}$$

and

$$T > \frac{1}{\sqrt{1 + \frac{(a\Omega)^2}{2}}}\left(\pi - \arcsin\left(\frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}}\right)\right) \tag{29}$$

On the other hand, to find the frequency of the limit cycle, we let $\theta = \omega_0 t + \phi$ denote the argument of the cosine in Eq. (12). Then the frequency of the periodic solution is

$$\omega = \frac{d\theta}{dt} = \omega_0 + \frac{d\phi}{dt} \tag{30}$$

Using Eq. (24) yields

$$\omega = \omega_0 - \frac{\lambda}{2\omega_0}\cos\omega_0 T - \frac{3\delta}{8\omega_0}(2(a\Omega)^2 + 1)R^2 \tag{31}$$

Eq. (31) gives a relationship between the frequency of the limit cycle ω, the FH excitation frequency Ω, with $\omega_0 = \sqrt{1 + \frac{(a\Omega)^2}{2}}$, and the delay period T. A condition for the existence of the limit cycle is guaranteed when the frequency ω is positive. In Eq. (31) the frequency of the limit cycle will be positive for all T in the (T, Ω) plane.

Note that the conditions (28) and (29) on time delay T, separating regions where self-excited oscillations take place from those where self-excited oscillations do not exist, are valid when the inequality $\frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}} \leqslant 1$ is held. This inequality requires $\alpha < \lambda$ which can be seen in Fig. 1 in which conditions (28) and (29) are plotted for different values of λ greater than $\alpha = 0.5$. These curves suggest that for a fixed value of the delay amplitude, there exists a maximal critical value Ω_c for which the region where self-excited oscillation can be eliminated appears as well as the corresponding time delay T_c. A relationship providing these critical values Ω_c and T_c can be obtained by using Eq. (27). This leads to

$$\Omega_c = \frac{1}{a}\sqrt{2\left(\frac{\lambda^2}{\alpha^2} - 1\right)} \tag{32}$$

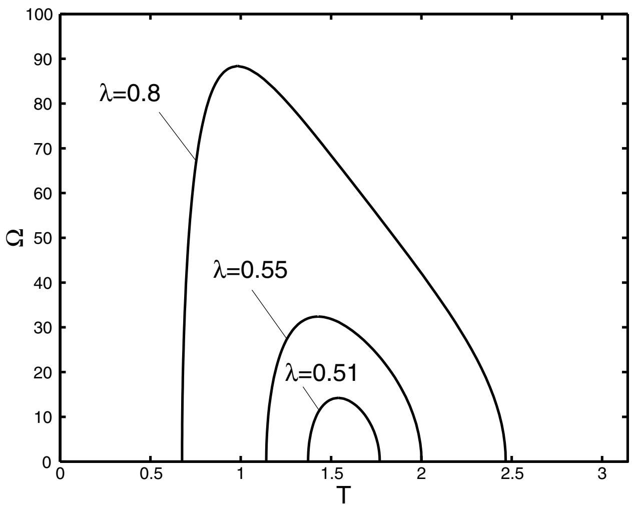

Fig. 1. Curves delimiting the existence regions of limit cycle, conditions (28) and (29), with parameter values $a = 0.02$, $\alpha = \beta = 0.5$.

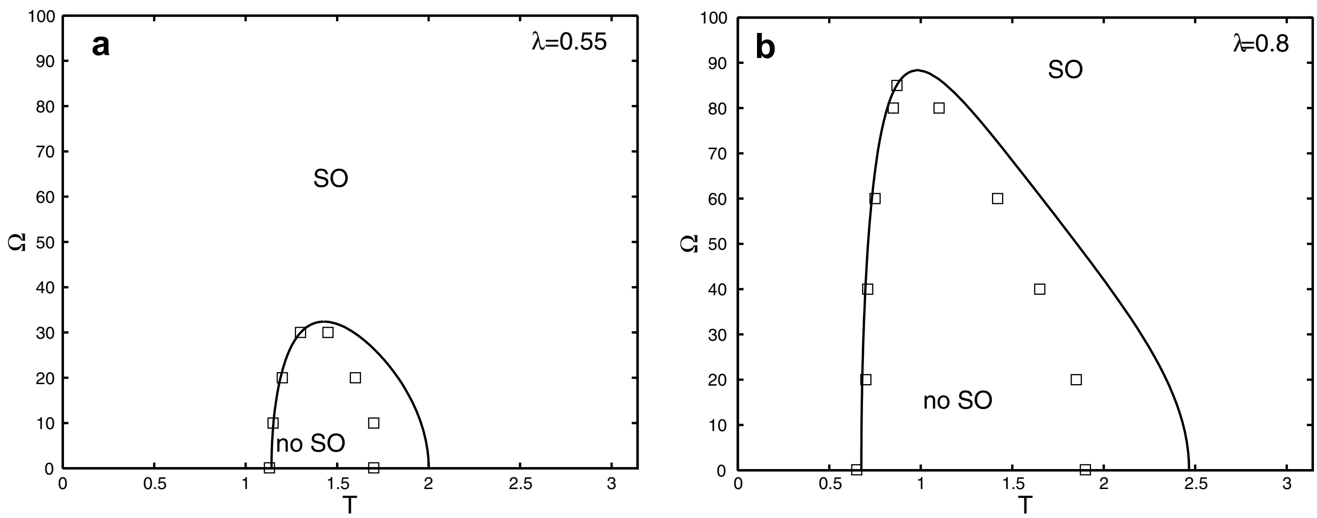

Fig. 2. Comparison between analytical results (solid line) based on conditions (28) and (29) and numerical integration (squares) of the original system, Eq. (2); $a = 0.02$, $\alpha = \beta = 0.5$, $\delta = 1/6$. SO: self-oscillations.

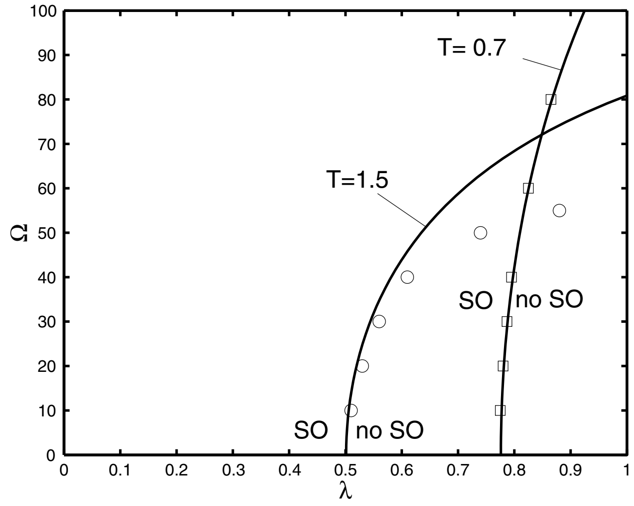

Fig. 3. Comparison between analytical results (solid line), Eq. (26), and numerical integration (squares) and (circles) of the original system, Eq. (2); $a = 0.02$, $\alpha = \beta = 0.5$, $\delta = 1/6$. SO: self-oscillations.

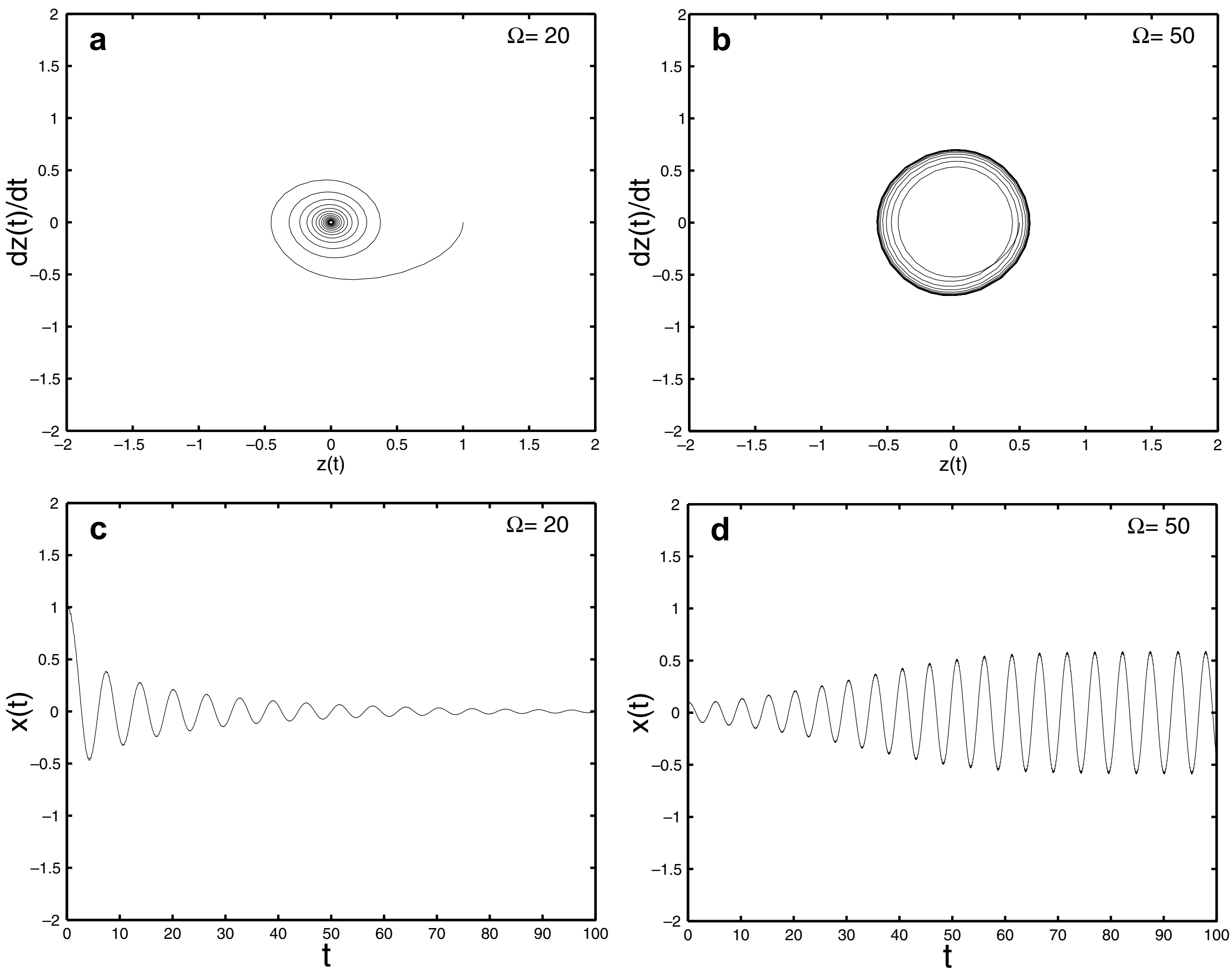

Fig. 4. Phase portraits of slow dynamic $z(t)$, Eq. (10), and time histories of the full motion $x(t)$, Eq. (2), with parameter values as for Fig. 2(a) and $T = 1.4$.

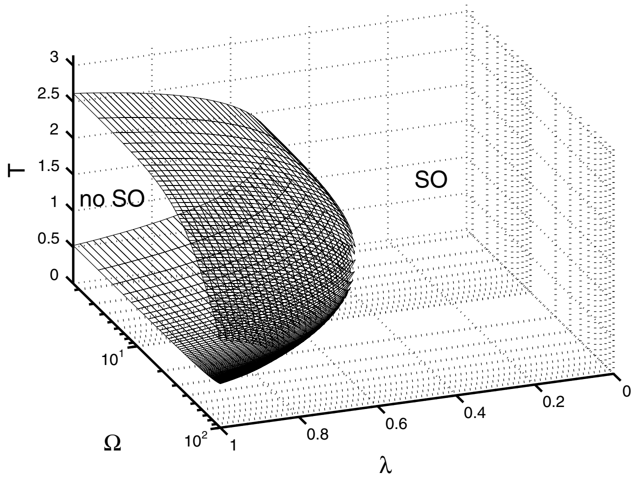

Fig. 5. Existence region of limit cycle, conditions (28) and (29); $a = 0.02$, $\alpha = \beta = 0.5$. SO: self-oscillations.

and

$$T_c = \frac{\alpha}{\lambda}\frac{\pi}{2} \tag{33}$$

From Fig. 1 we can see that the region where limit cycle can be eliminated increased by increasing the delay amplitude λ. A comparaison between analytical results (solid line), Eqs. (28) and (29), and numerical integration of the original system (squares), Eq. (2), of curves delimiting these regions is shown in Fig. 2. This numerical calculations are done in Matlab by using the integrating function dde23 [15]. Note that the conditions (28) and (29) providing the existence region of limit cycle does not depend upon the nonlinearity coefficient δ.

In Fig. 3 we show comparisons of the existence region of limit cycle in the (λ, Ω) plane for different values of time delay.

Phase portraits of slow dynamic $z(t)$, Eq. (10), and time histories of the full motion, Eq. (2), are shown in Fig. 4 for $T = 1.4$, $\lambda = 0.55$ and for different values of Ω (see Fig. 2a).

Finally, Fig. 5 shows the surface curve in three dimensional Ω-λ-T parameter space delimiting the region where limit cycle can be eliminated.

5. Conclusions

In this work we have investigated the interaction effect of vertical FH parametric excitation and time delay on self-excited vibrations in a van der Pol pendulum. We have used the method of direct partition of motion which provides the main equation governing the slow dynamic and then applied averaging method on the slow dynamic to derive the slow flow which is analyzed to analytically locate regions where self-excited vibrations can be eliminated. Numerical results were reported and compared to the analytical approximations showing a good agreement for small time delay. We have shown that in the case of a vertical parametric excitation, self-oscillations can take place only when the delay amplitude λ is greater than the damping coefficient α (Eq. (32)). In addition, our analysis showed that the conditions delimiting the region of existence of the limit cycle does not depend upon the nonlinearity coefficient δ. The most significant result of this work is the conclusion that vertical FH parametric excitation, in the presence of small time delay, can eliminate undesirable self-excited vibrations. Even for low values of the parametric frequency Ω_c, this suppression can be realized in the vicinity of the time delay T_c (Fig. 1). These self-excited vibrations, however, persist for all Ω in the case of a vertical parametric excitation without delay as shown in [6].

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